Nuclear Physics B196 (1982) 285–300

GENERAL METHOD FOR ANALYZING HIGGS POTENTIALS*

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Received 14 September 1981

We define a set of group-invariant orbit parameters. Using them, we describe a method for determining the energy and residual symmetry of the Higgs potential minimum. The method is general, and offers significant advantages in analyzing complicated Higgs potentials.

1. Introduction

After the discovery of the Higgs mechanism, it was applied to the unification of electromagnetic and weak interactions [1] with great success. Minimizing the potential was easy because the symmetry group and the representation of higgsons were small. But the subsequent appearance of grand unification theories [2] brought non-trivial difficulties to the symmetry-breaking problem. First, it became very difficult to find the minimum of the classical potential because of the huge size of the gauge group and the representations of higgsons. Secondly, the problem of gauge hierarchy, plus the sheer proliferation of Higgs parameters, raised a doubt whether the Higgs mechanism is a sensible way to break the symmetries at all.

Regardless of the outcome of the gauge hierarchy [3] and proliferation problems, the first difficulty is worth investigating in its own right because it appears in other branches of physics as well, and because an adequate assessment of the various grand unification theories requires a knowledge of their detailed consequences. We shall address ourselves to this first, essentially technical problem.

For the case of one irreducible representation of higgsons, much work has already been done. Michel [4] set the direction of research by focusing attention on invariant potentials and orbits instead of dealing with each component of the Higgs field. He proposed a fundamental conjecture and proved it for simple cases: if the representation of the symmetry group G of fourth degree Higgs potential is irreducible on the real, its minima preserve maximal little groups. Li [5] worked out some relatively simple cases using conventional methods with brilliant diagonalizing techniques. But his techniques could not be applied to multi-dimensional representations. For the

* Work supported in part by the US Department of Energy under contract no. DE-AC-03-76ER00068.

case of two irreducible representations, Gell-Mann and Slansky proposed a generalization of Michel's conjecture [6], which will be discussed in this paper. Some specific examples have been worked out for the cases of SU(n) adjoint + vector [7] and of SO(10) adjoint + spinor [8].

In the present paper we describe a general method which simplifies the analysis of complicated Higgs problems. Checking the Gell-Mann–Slansky conjecture is greatly facilitated. The examples considered by Li can be solved trivially. A simple and readily visualized treatment becomes possible for the more complicated quartic Higgs potential of a pair of irreducible representations, the adjoint + vector representations of SU(n) or SO(n) [9].

Our paper is organized as follows. In sect. 2 we briefly review the Higgs problem and define orbit parameters. In sect. 3 we describe our method of solving the Higgs problem by use of these parameters, and apply it to the case where there is one irreducible representation (irrep). In sect. 4 we apply the method to the case of two irreps. In sect. 5 we discuss the Gell-Mann–Slansky conjecture and make concluding comments. More specific examples will be given in forthcoming papers [9, 12].

2. Higgs problem and orbit parameters

Though our method can be applied to any kind of Higgs potential, we will take a rather simple case to show the main ideas.

In a non-abelian gauge theory, where the symmetry group is G × reflection and the higgsons belong to an n-dimensional irreducible representation $\underline{R}$ of G, the Higgs potential can be written as

$$V(\varphi)=-\tfrac{1}{2}m^2\sum_{i=1}^{n}\varphi_i^*\varphi_i+\tfrac{1}{4}A\left(\sum_{i=1}^{n}\varphi_i^*\varphi_i\right)^2$$

$$+\tfrac{1}{4}A_1 f_{ijkl}\varphi_i^*\varphi_j\varphi_k^*\varphi_l+\tfrac{1}{4}A_2 g_{ijkl}\varphi_i^*\varphi_j\varphi_k^*\varphi_l+\cdots\,. \tag{1}$$

$V(\varphi)$ is invariant under a group transformation

$$\varphi_j'=\sum_{i=1}^{n}T(\vartheta)_{ji}\varphi_i,$$

where $T(\vartheta)$ is an n-dimensional matrix corresponding to a group element. In general

$$T(\vartheta)=\exp\left[-i\sum_{i=1}^{N}\vartheta_L X_L\right],$$

where X_L are generators of the group and ϑ_L are group parameters specifying an element of the group.

As is well known, due to the negative mass term, the minimum of the potential occurs at some non-zero values v of φ. The vacuum, defined to be at the minimum of the potential, respects only a subgroup G′ of the symmetry group G of the lagrangian. Mathematically speaking, $T(\vartheta)v = v$ only if $T(\vartheta)$ is an element of $G' \subset G$, otherwise $T(\vartheta)v \neq v$.

When one tries to find the minimum of the potential, one faces significant difficulties:

(i) There are many components of φ and finding the solution for arbitrary coupling coefficients by setting $\partial V/\partial\varphi_i = 0$ is very difficult.

(ii) Even if we give special numerical values to the coefficients, it will not be helpful because the minimum occurs along valleys in φ space. To put it more clearly, we may choose $\varphi_1 = v$ and $\{X_1, X_2, X_3\}$ as our subgroup singlet and generators respectively, or we may equally well choose $\varphi_2 = v$ and $\{X_4, X_5, X_6\}$. In general there is a continuum of equivalent sets of φ_i and $\{X_L, X_M, X_N\}$. The Higgs potential is totally blind to such differences.

We will now introduce some useful group theoretical concepts. The orbit of φ_a is defined to be the set of states $\varphi^{(a)}$ that can be expressed as $\varphi^{(a)} = T(\vartheta)\varphi_a$ with $T(\vartheta)$ an element of G. It can easily be shown that all the states $\varphi^{(a)}$ on the orbit respect the same group, called the little group of the orbit, as φ_a does. If the $T(\vartheta)$ are unitary, then all the states $\varphi^{(a)}$ have the same norm $\varphi_a^*\varphi_a$. In general, there is a continuum of orbits respecting the same little group. The set of all such orbits is called the stratum of the little group. When we seek a solution to the Higgs problem, we are actually seeking the orbit that minimizes the potential, and its little group*.

Two important theorems concerning invariants and orbits can be found in the literature [10]:

Theorem 1: Invariant polynomials $P(\varphi)$ specify orbits of φ.

Theorem 2: There exists a basic set of invariant polynomials $I_a(\varphi)$ such that every invariant polynomial $P(\varphi)$ can be expressed as a polynomial of I_a: $P(\varphi) = \bar{P}[I_a(\varphi)]$. The number l of basic invariants is different for each different representation $\underline{R}$. We can visualize an orbit as a point in the l-dimensional space of I_a.

Our crucial observation is that the dimensionless ratios of invariants to the magnitude of the φ vector, for example

$$\lambda = f_{ijkl}\varphi_i^*\varphi_j\varphi_k^*\varphi_l \Big/ \left(\sum_{i=1}^{n} \varphi_i^*\varphi_i \right)^2 , \tag{2}$$

can also be used to specify strata, and yield a powerful tool in minimum problem. We will call the dimensionless ratios orbit parameters. They contain all the directional information and can be considered as a set of angles. Instead of pursuing individual components of the φ vector, we should direct our attention to these group invariant orbit parameters.

* In some cases there may be more than one stratum at the minimum energy.

3. One irreducible representation

When there is only one irreducible representation $\underline{R}$ of higgsons, the Higgs potential can be written as

$$V(\varphi) = -\tfrac{1}{2}m^2\|\varphi\| + \tfrac{1}{4}\|\varphi\|^2[A + A_1\lambda_1(\hat{\varphi}) + A_2\lambda_2(\hat{\varphi}) + \cdots], \tag{3}$$

where

$$\|\varphi\| = \sum_{i=1}^{n} \varphi_i^* \varphi_i, \qquad \hat{\varphi}_i = \varphi_i / \|\varphi\|^{1/2}.$$

Since we want the potential $V(\varphi)$ to increase to $+\infty$ as $\|\varphi\| \to \infty$, we impose a condition on the coupling coefficients:

$$A + A_1\lambda_1(\hat{\varphi}) + A_2\lambda_2(\hat{\varphi}) + \cdots > 0 \qquad \text{for any } \lambda_i(\hat{\varphi}). \tag{4}$$

Our new variables are $\|\varphi\|$ and $\lambda_i(\hat{\varphi})$. In spite of this bold reduction in the number of variables, we don't lose any generality because they contain all the information needed to determine the minimum of the potential.

If we choose a particular direction in φ space, then the orbit parameters will take definite values. We can easily see how the Higgs potential behaves in this direction:

$$V = -\tfrac{1}{2}m^2\|\varphi\| + \tfrac{1}{4}A'\|\varphi\|^2, \tag{5}$$

where m and A' are constant numbers. This function behaves like fig. 1.

The minimum for this particular choice of $\lambda_i(\hat{\varphi})$ is found by setting

$$\frac{\partial V}{\partial\|\varphi\|} = \tfrac{1}{2}[-m^2 + (A + A_1\lambda_1 + A_2\lambda_2 + \cdots)\|\varphi\|] \tag{6}$$

equal to zero. We obtain

$$\|\varphi\|_0 = \frac{m^2}{A + A_1\lambda_1 + A_2\lambda_2 + \cdots}, \tag{7}$$

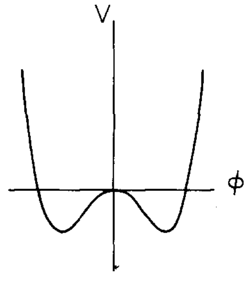

Fig. 1. General shape of a fourth degree Higgs potential for one irrep.

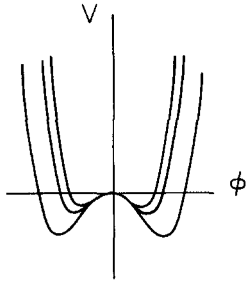

Fig. 2. The location of the directional minimum moves around as we change the direction in φ space.

which is automatically positive for $m^2 > 0$ due to condition (4). At the extremum,

$$V_0(\varphi) = V(\varphi)|_{\|\varphi\| = \|\varphi\|_0}$$

$$= -\frac{1}{4}\frac{m^4}{(A + A_1\lambda_1 + A_2\lambda_2 + \cdots)}$$

$$= -\tfrac{1}{4}m^2\|\varphi\|_0. \tag{8}$$

As we change the direction in φ space [i.e., the $\lambda_i(\hat{\varphi})$], the location of the minimum will move around as in fig. 2. To find the absolute minimum we just have to look for the lowest of those directional minima. Since

$$\frac{\partial V}{\partial \lambda_i} = \tfrac{1}{4}\|\varphi\|^2 A_i, \tag{9}$$

V is a monotonic function of λ_i. Thus the absolute minimum of V is *not* at $\partial V/\partial\lambda_i = 0$, but at the boundary points of the region of physical λ_i.

To find these boundary points, note that the orbit parameters are dimensionless ratios of invariants such as eq. (2), i.e., they depend on $\hat{\varphi}$ whose magnitudes are less than one. These defining equations permit a precise determination of the region of physical λ_i. In particular it is immediately clear that for any configuration of $\hat{\varphi}$, the range of λ_i is bounded above and below:

$$\lambda_{i\min} \leq \lambda_i(\hat{\varphi}) \leq \lambda_{i\max}.$$

In an actual calculation the first practical task will be to calculate the physical region of $\lambda_i(\hat{\varphi})$, which we shall call the orbit space*.

* A more accurate description is the stratum space, which is a projection of the true orbit space.

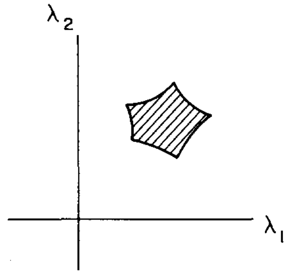

Fig. 3. Qualitative form of a two-dimensional orbit space.

Suppose there are two orbit parameters λ_1 and λ_2. In the λ_1-λ_2 plane, the orbit space will look something like fig. 3. It is important to note [viz., eq. (2)] that the orbit space is independent of Higgs coupling coefficients and masses, though it does depend on the group and the representation.

Turning our attention to the potential, let us put

$$C = A + A_1\lambda_1 + A_2\lambda_2. \tag{10}$$

For given values of A, A_1, A_2 and C, this will represent a line in λ_1-λ_2 space (fig. 4). According to condition (4) the line can only intersect orbit space when $C > 0$. As we increase C at fixed A, A_1, A_2, the line will sweep the orbit space. The minimum physical value of C will occur where the line first touches the orbit space. By eq. (8) this corresponds to the absolute minimum of the Higgs potential. Above the absolute minimum of V there is a continuous range of V and $\|\varphi\|_0$ where $\partial V/\partial\|\varphi\| = 0$ can be satisfied by some choice of λ_i. As we further increase C, the line finally leaves the orbit space at the highest of the directional minima where V has the form of the upper curve in fig. 2.

Our considerations with fig. 2 have suggested that V has no other extremum than the minimum at $\|\varphi\|_0$ (distributed over a range of $\|\varphi\|_0$ as the λ_i vary) and the local

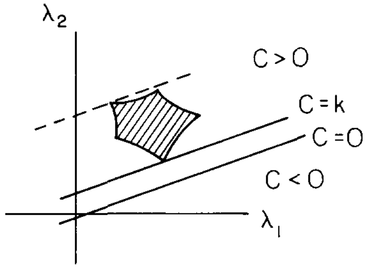

Fig. 4. The procedure for finding the absolute minimum of a Higgs potential with two orbit parameters for one irrep.

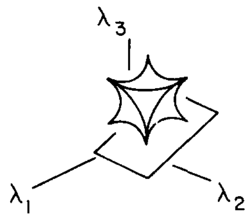

Fig. 5. Plane touching a three-dimensional orbit space.

maximum at $\|\varphi\| = 0$. These statements are verified by noting that

$$\frac{\partial^2 V}{\partial \|\varphi\|^2} = \tfrac{1}{2}[A + A_1\lambda_1 + A_2\lambda_2 + \cdots] \tag{11}$$

is always positive due to condition (4).

For some special values of A_1 and A_2 the line may first touch the boundary of orbit space at two points. In such cases there will be two different valleys of extrema (two orbits) that can not be connected by a gauge transformation.

If there are more than two orbit parameters, then

$$C = A + A_1\lambda_1 + A_2\lambda_2 + \cdots + A_s\lambda_s \tag{12}$$

represents a plane in λ space and the situation can be depicted as in fig. 5. The procedure to find the minimum will be the same as before. Since the absolute minimum always occurs at the boundary of the orbit space, we have to find the $(s-1)$ surface parameters and the value of the potential at the first contact point in s-dimensional orbit parameter space*.

When the representation is complex, the potential can in general contain terms of the type

$$(Hh_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l + \text{complex conjugate}) = 2|H|\cos[h + \vartheta(\hat{\varphi})]\,\eta(\hat{\varphi})\|\varphi\|^2, \tag{13}$$

where

$$|H| = \text{magnitude of } H,$$

$$h = \text{argument of } H,$$

$$0 \leqslant \eta(\hat{\varphi}) \leqslant \eta_{\max}.$$

* Due to the restriction to quadratic and quartic potentials, we are dealing with a subspace of the complete $(l-1)$-dimensional orbit parameter space.

At the minimum of the potential, $h+\vartheta=\pi$, and $\eta(\hat{\varphi})$ is determined in the same way as the λ_i.

In problems where cubic potentials or effective potentials are considered, the potential takes the general form

$$V(\varphi)=\tfrac{1}{2}A\|\varphi\|+\tfrac{1}{3}\left[B_1\beta_1(\hat{\varphi})+B_2\beta_2(\hat{\varphi})+\cdots\right]\|\varphi\|^{3/2}$$
$$+\tfrac{1}{4}\left[C+C_1\gamma_1(\hat{\varphi})+\cdots\right]\|\varphi\|^2+\tfrac{1}{5}\left[D_1\delta_1(\hat{\varphi})+\cdots\right]\|\varphi\|^{5/2}+\cdots. \quad (14)$$

If we choose a direction in φ space, all the orbit parameters will become constant numbers and we can easily see how the function behaves in that direction of φ. The function will always be monotonic with respect to orbit parameters. However, due to theorem 2, some orbit parameters associated with higher degree ($\geqslant 5$) polynomial invariants are polynomials of lower degree parameters. Eventually the problem will become non-linear and detailed solution will be far more complicated. We may keep the linearity formally by treating all the orbit parameters as independent and introducing non-linear constraints among them.

The classic paper of Li dealt with single irreps of higgsons in SU(n) and SO(n). Only cases involving a single orbit parameter were considered, i.e., the orbit space was always a line $\lambda_{\min}\leqslant\lambda(\hat{\varphi})\leqslant\lambda_{\max}$. The present paper opens the way to analysis of more complex cases involving a single irrep.

4. Two irreducible representations

When there are two irreps, $\underline{R}$ and $\underline{S}$, of higgsons φ and χ the most general renormalizable Higgs potential invariant under G $\times$ reflection can be written as

$$V(\varphi,\chi)=-\tfrac{1}{2}M^2\|\varphi\|-\tfrac{1}{2}m^2\|\chi\|$$
$$+\tfrac{1}{4}\left[A+A_1\alpha_1(\hat{\varphi})+A_2\alpha_2(\hat{\varphi})+\cdots\right]\|\varphi\|^2$$
$$+\tfrac{1}{4}\left[C+C_1\gamma_1(\hat{\chi})+C_2\gamma_2(\hat{\chi})+\cdots\right]\|\chi\|^2$$
$$+\tfrac{1}{2}\left[B+B_1\beta_1(\hat{\varphi},\hat{\chi})+\cdots\right]\|\varphi\|\|\chi\|. \quad (15)$$

While α_i and γ_i specify the orbits and associated little groups of $\underline{R}$ and $\underline{S}$, respectively, β_i will specify relative directions between orbits of $\underline{R}$ and orbits of $\underline{S}$. If χ moves on an orbit with the direction of φ fixed, the little group of the reducible representation ($\underline{R}+\underline{S}$) will change whereas the separate little groups of $\underline{R}$ and $\underline{S}$ remain the same. β_i will specify the location of χ on its orbit.

Let us define

$$A' \equiv A + A_1\alpha_1(\hat{\varphi}) + A_2\alpha_2(\hat{\varphi}) + \cdots ,$$

$$C' \equiv C + C_1\gamma_1(\hat{\chi}) + C_2\gamma_2(\hat{\chi}) + \cdots ,$$

$$B' \equiv B + B_1\beta_1(\hat{\varphi}, \hat{\chi}) + \cdots . \tag{16}$$

Again we will impose positivity conditions on coupling coefficients so that $V \to +\infty$ as $\|\varphi\| \to \infty$ and/or $\|\chi\| \to \infty$:

$$A' > 0, \qquad C' > 0, \qquad B' > -\sqrt{A'C'} . \tag{17}$$

We will treat $\|\varphi\|, \|\chi\|, \alpha_i(\hat{\varphi}), \gamma_i(\hat{\chi})$, and $\beta_i(\hat{\varphi}, \hat{\chi})$ as independent variables and extremize the potential with respect to these. The reasoning is similar to the one irrep case. If we choose a particular direction in φ-χ space, all the orbit parameters will be determined and the potential reduces to a function of $\|\varphi\|$ and $\|\chi\|$:

$$V = -\tfrac{1}{2}M^2\|\varphi\| - \tfrac{1}{2}m^2\|\chi\| + \tfrac{1}{4}A'\|\varphi\|^2 + \tfrac{1}{4}C'\|\chi\|^2 + \tfrac{1}{2}B'\|\varphi\|\|\chi\| . \tag{18}$$

This function behaves like fig. 6.

The extremum for the particular choice of orbit parameters, conveniently expressed in terms of the variables $r \equiv \|\varphi\|^{1/2}$ and $s \equiv \|\chi\|^{1/2}$, is given by the conditions

$$\frac{\partial V}{\partial r} = r(A'r^2 + B's^2 - M^2) = 0, \qquad \frac{\partial V}{\partial s} = s(B'r^2 + C's^2 - m^2) = 0. \tag{19}$$

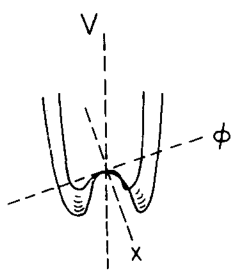

Fig. 6. General shape of a fourth degree Higgs potential for two irreps.

There are four solutions;

$$\text{(I)} \qquad r=s=0, \tag{20a}$$

$$\text{(II)} \qquad r=0, \qquad s^2=m^2/C', \tag{20b}$$

$$\text{(III)} \qquad s=0, \qquad r^2=M^2/A', \tag{20c}$$

$$\text{(IV)} \qquad r^2=\|\varphi\|_0=\frac{M^2C'-m^2B'}{A'C'-B'^2}, \qquad s^2=\|\chi\|_0=\frac{m^2A'-M^2B'}{A'C'-B'^2}. \tag{20d}$$

To ascertain which solution is the minimum for this particular choice of direction in φ-χ space (which we shall refer to as the "directional minimum"), recall that at a minimum the second derivatives

$$\frac{\partial^2 V}{\partial r^2}=(A'r^2+B's^2-M^2)+2A'r^2, \qquad \frac{\partial^2 V}{\partial s^2}=(B'r^2+C's^2-m^2)+2C's^2,$$

$$\frac{\partial^2 V}{\partial r\partial s}=2B'rs, \tag{21}$$

must satisfy

$$\frac{\partial^2 V}{\partial r^2}>0, \tag{22a}$$

$$\frac{\partial^2 V}{\partial s^2}>0, \tag{22b}$$

$$\frac{\partial^2 V}{\partial r^2}\frac{\partial^2 V}{\partial s^2}>\left(\frac{\partial^2 V}{\partial r\partial s}\right)^2. \tag{22c}$$

Of course solution I is not a minimum unless $M^2<0$ and $m^2<0$, a case we shall not be concerned with. We see from eqs. (20)–(22) that solution II (pure χ) is a directional minimum if

$$m^2>0, \qquad m^2B'>M^2C'. \tag{23}$$

Solution III (pure φ) is a directional minimum if

$$M^2>0, \qquad M^2B'>m^2A'. \tag{24}$$

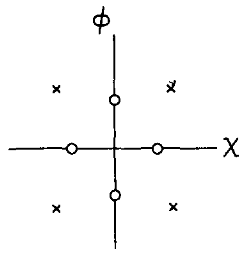

Fig. 7. Typical locations of directional minima (×) and saddle points (○).

Solution IV is a directional minimum if

$$M^2C' > m^2B', \tag{25a}$$

$$m^2A' > M^2B', \tag{25b}$$

$$A'C' > (B')^2, \tag{25c}$$

for $A' > 0$ and $C' > 0$.

$\|\varphi\|_0 > 0$ and $\|\chi\|_0 > 0$ is guaranteed only if conditions (25) are satisfied*. This is in contrast to the case of one irrep where $\|\varphi\|_0 \geqslant 0$ was ensured by the positivity conditions, $A' > 0$ and $m^2 > 0$. Relations (25) serve to replace the conditions $M^2 > 0$, $m^2 > 0$ which are overly strict because the Higgs fields φ and χ can both develop non-zero vacuum expectation values even with $M^2 < 0$ or $m^2 < 0$ when $B' < 0$.

From another point of view $M^2C' = m^2B'$ and $m^2A' = M^2B'$ represent the boundaries where the directional minimum shifts from solution IV to solution II or III respectively. If solution IV is the directional minimum, extrema II and III are saddle points (assuming now $m^2 > 0, M^2 > 0$) as indicated in fig. 7. In this case evaluation of the potential at the minimum yields

$$V_0(\hat{\varphi}, \hat{\chi}) = -\frac{1}{4}\frac{(m^4A' + M^4C' - 2M^2m^2B')}{(A'C' - B'^2)}$$

$$= -\tfrac{1}{4}\left(M^2\|\varphi\|_0 + m^2\|\chi\|_0\right). \tag{26}$$

When solution IV does not satisfy the conditions (25), it occupies a saddle point and either solution II (with $V_0 = -m^4/4C'$) or III (with $V_0 = -M^4/4A'$) becomes the directional minimum.

* Conditions (25) are only necessary conditions for solution IV to be the absolute minimum. There will be additional conditions for sufficiency [9].

The foregoing discussion has been concerned with a particular direction in φ-χ space. As we now change the direction in φ-χ space (i.e., the α_i, β_i and γ_i), the location of the minimum will move around. The absolute minimum will be the lowest of these directional minima. Since

$$\frac{\partial V}{\partial \alpha_i} = \tfrac{1}{4}\|\varphi\|^2 A_i, \tag{27a}$$

$$\frac{\partial V}{\partial \gamma_i} = \tfrac{1}{4}\|\chi\|^2 C_i, \tag{27b}$$

$$\frac{\partial V}{\partial \beta_i} = \tfrac{1}{2}\|\varphi\|\|\chi\| B_i, \tag{27c}$$

V is a monotonic function of the orbit parameters α_i, β_i, and γ_i. Thus, once again the absolute minimum of V occurs at a boundary of the orbit space rather than at $\partial V/\partial \alpha_i = 0$, etc.

To illustrate how determination of the absolute minimum proceeds, let us look into the simple case where

$$A' = A + A_1\alpha(\hat{\varphi}), \qquad C' = C + C_1\gamma(\hat{\chi}), \qquad B' = B + B_1\beta(\hat{\varphi}, \hat{\chi}). \tag{28}$$

Let us set

$$V_0(\hat{\varphi}, \hat{\chi}) = -\tfrac{1}{4}k. \tag{29}$$

Then from eq. (26),

$$m^4(A + A_1\alpha) + M^4(C + C_1\gamma) - 2M^2m^2(B + B_1\beta)$$

$$= k\left[(A + A_1\alpha)(C + C_1\gamma) - (B + B_1\beta)^2\right]. \tag{30}$$

It can easily be shown that the above equation represents a cone in $\alpha - \gamma - \beta$ space. After some coordinate transformations, it reduces to

$$Z^2 = \frac{A_1C_1}{2B_1^2}(X^2 - Y^2), \tag{31}$$

where

$$X = (X' + Y')/\sqrt{2}, \qquad X' = \alpha + \frac{A}{A_1} - \frac{M^4}{kA_1},$$

$$Y = (X' - Y')/\sqrt{2}, \qquad Y' = \gamma + \frac{C}{C_1} - \frac{m^4}{kC_1},$$

$$Z = \beta + \frac{B}{B_1} - \frac{M^2m^2}{kB_1}. \tag{32}$$

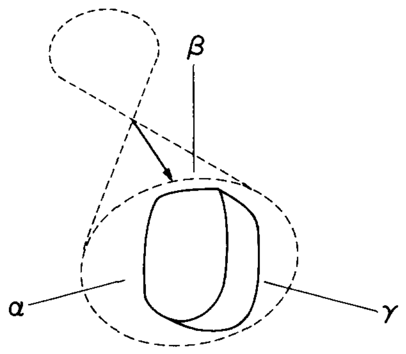

Fig. 8. The procedure for finding the absolute minimum of a Higgs potential with three orbit parameters α, β, and γ for two irreps. Pictured are the cone, the generating line along which it slides, and the irregular shaped orbit space.

While the coupling coefficients determine the shape and orientation of the cone, the value of k determines the location of the vertex of the cone which moves on a straight line in α-γ-β space as k varies. As we decrease k from $+\infty$, the cone begins to touch the orbit space at some k (fig. 8). This k gives the minimum energy, and the point of contact gives the orbit.

Some further details concerning the cone are as follows.

(i) The straight line along which the vertex of the cone moves lies on the cone (i.e., it is a generating line of the cone.).

(ii) The condition $A'C' = B'^2$ holds on the $k = \infty$ cone and $A'C' > B'^2$ holds inside it. Recalling that $A'C' > B'^2$ is a condition for solution IV (i.e., $\|\varphi\|_0$ and $\|\chi\|_0$ both non-zero), we see that when this solution gives the minimum energy, the orbit space lies entirely within the "forward" part of the cone, i.e., the part which narrows as k decreases (fig. 8).

(iii) The line along which the vertex moves is also the intersection of the two planes

$$M^2C' = m^2B'$$

and

$$m^2A' = M^2B' ,$$

which formed the boundary between solutions II or III and IV. These planes slice the inside of the cone into three pieces. Only when the cone touches the orbit space on the $M^2C' > m^2B'$, $m^2A' > M^2B'$ side of these planes do we get type IV solutions. Such type IV solutions yield the absolute minimum energy if they occur at $k > M^4/A'_0$ and $k > m^4/C'_0$.

While the formalism in the preceding two paragraphs is universal to all the cases where there are three orbit parameters α, γ, and β, each different case will have a different orbit space and different physical meaning for the boundary surface.

Of the two-irrep cases analyzed in the literature thus far, $(\overline{45}+\overline{16})$ of SO(10) [8] is of the above form with three orbit parameters whereas adjoint + vector representations of SO(n) or of SU(n) have only α and β. With our present machinery two-dimensional orbit space becomes relatively easy to analyze, and we shall work out the example of $(\underline{24}+\underline{5})$ of SU(5) in a forthcoming paper [9].

5. The little group at the absolute minimum

First consider the case where there is a single irrep $\underline{R}$ of a symmetry group G. $\underline{R}$ can be decomposed into a sum of irreps of a subgroup $G' \subset G$:

$$R = r_1 + r_2 + \cdots .$$

If $\underline{R}$ contains one singlet of G′, the stratum with little group G′ will be represented as a point in the orbit space. If $\underline{R}$ contains two singlets of G′, the corresponding stratum will normally be a curve in the orbit space, though it may still be a point in some cases. If $\underline{R}$ contains three singlets of G′, the stratum is likely to occupy a two-dimensional surface in the orbit space. In practice $\underline{R}$ tends to contain one singlet of the maximal little group and rarely two singlets.

Since the plane $C = A + A_1\lambda_1 + A_2\lambda_2 + \cdots$ just contacts the boundary surface of the orbit space at the minimum of the potential and there is no other condition than the positivity conditions, we are led to the following picture. If Michel's conjecture is true, then there will be cusps on the boundary of the orbit space corresponding to maximal little groups that yield one singlet, and there may be convex sections corresponding to maximal little groups that yield two or more singlets. Any section of the boundary that does not correspond to a maximal little group must be non-convex. Unfortunately, we have not yet been able to find a simple example where there are two orbit parameters to demonstrate the above geometry.

If we include higher degree polynomials, then polynomials of basic invariants occur. Since the potential is no longer monotonic with respect to orbit parameters, we have to find the minimum by solving $\partial V/\partial\lambda_i = 0$ and $\partial V/\partial\|\varphi\| = 0$. The extrema may occur inside the orbit space. Michel's conjecture will no longer hold in this situation. We strongly suppose that the criterion for the validity of Michel's conjecture is the linearity of the potential with respect to orbit parameters. We shall show that this is true in the examples of SU(n) adjoint [12].

Now, let us consider the case where there are two irreps, $\underline{R}$ and $\underline{S}$. Suppose the branching rules for these representations under $G' \subset G$ are

$$R = r_1 + r_2 + \cdots ,$$

$$S = s_1 + s_2 + \cdots .$$

If $\underline{R}$ contains one singlet and $\underline{S}$ one singlet of G′, then the stratum will be a point in the orbit space. If $\underline{R}$ contains one singlet and $\underline{S}$ two singlets of G′ or vice versa, the stratum will normally be a curve in the orbit space, though there are exceptions. If $\underline{R}$ contains one singlet and $\underline{S}$ three singlets of G′ or vice versa, the stratum is likely to be a two-dimensional surface. If $\underline{R}$ contains three singlets and $\underline{S}$ two singlets of G′ or vice versa, the stratum is likely to occupy a three-dimensional volume.

We are now ready to state the Gell-Mann–Slansky conjecture:

Suppose there are two irreps $\underline{R}$ and $\underline{S}$. First, we construct a list of maximal little groups and branching rules for* $\underline{R}$:

$$\begin{aligned} R &= 1 + r_1 + r_2 + \cdots \qquad && \text{for } G'_a \subset G \\ &= 1 + r_3 + r_4 + \cdots && \text{for } G'_b \subset G \\ &= \cdots \end{aligned}$$

For each G'_i the branching rules of $\underline{S}$ will be

$$\begin{aligned} S &= s_1 + s_2 + s_3 + \cdots \qquad && \text{for } G'_a \subset G \\ &= s_4 + s_5 + s_6 + \cdots && \text{for } G'_b \subset G \\ &= \cdots . \end{aligned}$$

Then we make a list of maximal little groups and branching rules for each s_i:

$$\begin{aligned} s_1 &= 1 + t_1 + t_2 + \cdots \qquad && \text{for } G^{(1)}_\alpha \subset G'_a \\ &= \cdots , \\ s_2 &= 1 + t_3 + t_4 + \cdots && \text{for } G^{(2)}_\beta \subset G'_a \\ &= \cdots , \\ s_4 &= 1 + t_5 + t_6 + \cdots && \text{for } G^{(4)}_\gamma \subset G'_b \\ &= \cdots , \\ s_5 &= 1 + t_7 + t_8 + \cdots && \text{for } G^{(5)}_\delta \subset G'_b \\ &= \cdots , \\ &= \cdots . \end{aligned}$$

* G_a is a maximal little group if no other little group includes it as a subgroup.

The conjecture states that the minimum of the Higgs potential will preserve no smaller subgroup than is in the list $\{G_\alpha^{(1)},\dots;\ G_\beta^{(2)},\dots;\ G_\gamma^{(4)},\dots;\ G_\delta^{(5)},\dots;\ \cdots\}$, called maxi-maximal little groups.

If their conjecture is to be true, then portions of the boundary must correspond to maxi-maximal little groups and the other portions corresponding to little groups smaller than maxi-maximal little groups must be avoided. In future papers we shall show that this is exactly true in the examples of SU(n) adjoint + vector [9].

For similar reasons as in the one irrep case, we suppose that the Gell-Mann–Slansky conjecture holds when the potential is linear with respect to orbit parameters.

It is an interesting problem for the future whether the concrete approach of the present paper can be given a deeper mathematical understanding. Such hope is found in Mumford's book [11] whose existence was known to us in tracing back references given in Abud and Sartori's recent paper [11].

I would like to thank Prof. M. Gell-Mann and Prof. S. Frautschi for many helpful discussions.

References

[1] S. Weinberg, Phys. Rev. Lett. 19 (1967) 1264;
A. Salam, Elementary particle theory, ed. N. Svartholm (Almqvist and Forlag, Stockholm, 1968)
[2] H. Georgi and S.L. Glashow, Phys. Rev. Lett. 32 (1974) 438;
H. Fritzsch and P. Minkowski, Ann. of Phys. 93 (1975) 193
[3] H. Georgi, H.R. Quinn and S. Weinberg, Phys. Rev. Lett. 33 (1974) 451;
S. Weinberg, Phys. Lett. 82B (1979) 387;
J. Ellis, M.K. Gaillard, A. Petermann and C. Sachrajda, Nucl. Phys. B164 (1980) 253
[4] L. Michel, Lecture Notes in Physics 6 (Springer-Verlag, 1970) p. 36;
L. Michel and L.A. Radicati, Ann. of Phys. 66 (1971) 758;
L. Michel, CERN TH 2716 (1979)
[5] L.-F. Li, Phys. Rev. D9 (1974) 1723
[6] M. Gell-Mann, private communication (1980);
R. Slansky, Phys. Reports (1981)
[7] M. Magg and Q. Shafi, Z. Phys. C4 (1980) 63;
F. Buccella, H. Ruegg and C.A. Savoy, Nucl. Phys. B169 (1980) 68
[8] F. Buccella, H. Ruegg and C.A. Savoy, Phys. Lett. 94B (1980) 491
[9] S. Frautschi and J.S. Kim, Nucl. Phys. B196 (1982) 301;
J.S. Kim, SO(N) Higgs problem with adjoint + vector representations, Cal Tech preprint (Dec., 1981)
[10] H. Weyl, Classical groups, 2nd ed. (Princeton Univ. Press, 1946);
J.A. Dieudonne and J.B. Carrel, Adv. in Math. 4 (1970) 1;
D. Luna, Ann. Inst. Fourier 26 (1976) 33
[11] D. Mumford, Geometric invariant theory, Erg. der Math. 34 (1965);
M. Abud and G. Sartori, The geometry of orbit-space and natural minima of Higgs potentials, Padova Univ. preprint IFPD 45/81 (Mar., 1981)
[12] J.S. Kim, to appear in Nucl. Phys. B